

Monotonicity of the polaron energy II: General theory of operator monotonicity

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Abstract

We construct a general theory of operator monotonicity and apply it to the Fröhlich polaron hamiltonian. This general theory provides a consistent viewpoint of the Fröhlich model.

1 Introduction

This paper is a sequel to [31]. To explain our motivation of this work, let us recall the result in [31] first. In the previous work [31], we studied the Fröhlich hamiltonian defined by

$$H_\Lambda = -\frac{1}{2}\Delta_x - \sqrt{\alpha}\lambda_0 \int_{|k|\leq\Lambda} dk \frac{1}{|k|} [e^{ik\cdot x}a(k) + e^{-ik\cdot x}a(k)^*] + N_f. \quad (1.1)$$

Here Λ is the ultraviolet cutoff. (The complete definition of H_Λ will be recalled in the subsequent section.) One of the main results in [31] is stated as follows.

Theorem 1.1 [31] *Let $E_\Lambda = \inf \text{spec}(H_\Lambda)$. Then E_Λ is strictly decreasing in Λ .*

In the proof of Theorem 1.1, a new operator monotonicity played important roles. To illustrate the meaning of our operator monotonicity, we introduce some terms. Let \mathfrak{p} be a convex cone in the Hilbert space \mathfrak{h} . \mathfrak{p} is called to be *self-dual* if it satisfies

$$\mathfrak{p} = \{x \in \mathfrak{h} \mid \langle x, y \rangle \geq 0 \forall y \in \mathfrak{p}\}. \quad (1.2)$$

Let A, B be linear operators in \mathfrak{h} . For simplicity, suppose both are bounded. If A and B satisfy $(A - B)\mathfrak{p} \subseteq \mathfrak{p}$, we denote this as $A \supseteq B$ w.r.t. \mathfrak{p} . It is easily checked that the binary relation “ \supseteq ” is a partial order. Thus we can regard the relation $A \supseteq B$ w.r.t. \mathfrak{p} as an operator inequality. The operator monotonicity, our central subject in this paper, is expressed as follows. Let $\{A_s\}_{s \geq 0}$ be a family of operators in \mathfrak{h} . We say A_s is *monotonically decreasing* if

$$s > s' \implies A_{s'} \supseteq A_s \text{ w.r.t. } \mathfrak{p}. \quad (1.3)$$

To prove Theorem 1.1, we effectively applied this notion in [31], namely, we selected a proper self-dual cone so that a family of hamiltonians $\{H_\Lambda\}_{\Lambda \geq 0}$ becomes monotonically

decreasing under the choice. (It will be seen that we can extend the notion of operator monotonicity to unbounded operators, see Section 2.) Combining this monotonicity and a general theorem established in [31], one obtained the assertion in Theorem 1.1. In this paper, we search for further possibilities of the operator monotonicity. We construct a theory of the singular perturbation by the operator monotonicity, and apply it to the Fröhlich polaron hamiltonian. The results in the part I[31] are included in this theory. Our theory in this paper provides a consistent viewpoint of the polaron as well.

Next we briefly background mathematical difficulties studying the Fröhlich polaron model and then state the results concerned these difficulties. When we tackle the problem of investigating the mathematical aspects of the polaron model, we unavoidably encounter the question how we define the hamiltonian. In various physical literatures, one presupposes that the hamiltonian without the ultraviolet cutoff $H_{\Lambda=\infty}$ is given. However it is not obvious whether the hamiltonian $H_{\Lambda=\infty}$ is definable or not. To clarify the point, put $\Lambda = \infty$ in the electron-phonon interaction term in (1.1). Then since $1/|k|$ is not in $L^2(\mathbb{R}^3)$, the interaction term can not be defined as a linear operator in the Hilbert space. (Remark the symbols $\int_{\mathbb{R}^3} dk f(k)a(k)$ and $\int_{\mathbb{R}^3} dk f(k)a(k)^*$ are well-defined linear operators only if $f \in L^2(\mathbb{R}^3)$.) If one looks at only interaction term, the interaction term does not have mathematical meaning at $\Lambda = \infty$, however if our vision broaden and we study the whole hamiltonian including not only the electron-phonon interaction term but also the electron and phonon kinetic energy terms, the limit $\lim_{\Lambda \rightarrow \infty} H_{\Lambda}$ exists in a suitable sense. This is the standard way how to define the Fröhlich hamiltonian [8, 15, 16, 17, 21, 35, 36, 39]. In this paper, we prove the existence of the limiting hamiltonian by applying the general theory of the operator monotonicity. As far as we know, our proof is novel. In addition, this provides an important example of our operator monotonicity.

Next problem is to study the ground state property of the hamiltonian without the ultraviolet cutoff. (Here keep in mind that we are discussing the ground state of the hamiltonian at a fixed total momentum now.) Remark that existence of the ground state has been already established [15, 17, 34, 41]. In this paper we argue the uniqueness of the ground state. This is known to be rather difficult because the hamiltonian is defined through the limiting procedures. In [29], the author proved the uniqueness of the ground state. In the present work, we will give a different proof by the general theory of the operator monotonicity.

Once again, let us reconfirm the aim of this work. Through analysis of the Fröhlich hamiltonian, we build a general theory of the operator monotonicity. We expect the further validity of our operator inequalities is revealed through our attempt.

The organization of the paper is as follows. In Sects. 2 and 3, we establish a general theory of our operator monotonicity. Terminologies from the self-dual cone analysis [29, 30, 31, 32, 33] are introduced in this section. In Sect. 4, we summarize basic facts of the second quantization. Sect. 5 is devoted to display applications of the general theory in Sects. 2 and 3 to the Fröhlich polaron model. In Sects. 6-10, we give detailed proofs of the results stated in Sect 5.

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2 A general theory of operator monotonicity

In this section, we will provide an abstract theory of operator monotonicity. As we will see in the later sections, this abstract theory has an important application to the condensed matter physics.

2.1 Definitions

To express our ideas, results and proofs, we need to introduce some technical terms from an earlier work [31]. The terms defined here serve as a language in this paper.

Let \mathfrak{h} be a complex Hilbert space and \mathfrak{p} be a convex cone in \mathfrak{h} . Then \mathfrak{p} is called to be *self-dual* if

$$\mathfrak{p} = \{x \in \mathfrak{h} \mid \langle x, y \rangle \geq 0 \ \forall y \in \mathfrak{p}\}. \quad (2.1)$$

Henceforth \mathfrak{p} always denotes the self-dual cone in \mathfrak{h} . The following properties of \mathfrak{p} are well-known [4, 22]:

Proposition 2.1 *One has the following.*

- (i) $\mathfrak{p} \cap (-\mathfrak{p}) = \{0\}$.
- (ii) *There exists a unique involution j in \mathfrak{h} such that $jx = x$ for all $x \in \mathfrak{p}$.*
- (iii) *Each element $x \in \mathfrak{h}$ with $jx = x$ has a unique decomposition $x = x_+ - x_-$, where $x_+, x_- \in \mathfrak{p}$ and $\langle x_+, x_- \rangle = 0$.*
- (iv) \mathfrak{h} is linearly spanned by \mathfrak{p} .

Remark 2.2 We clarify the precise meaning of (iv) in Proposition 2.1. Each $x \in \mathfrak{h}$ can be expressed as $x = \Re x + i\Im x$, where $\Re x = \frac{1}{2}(\mathbb{1} + j)x$ and $\Im x = \frac{1}{2i}(\mathbb{1} - j)x$. Clearly $j\Re x = \Re x$ and $j\Im x = \Im x$. Thus, by (iii), we have a unique decomposition $\Re x = (\Re x)_+ - (\Re x)_-$ with $\langle (\Re x)_+, (\Re x)_- \rangle = 0$. Similar property holds for $\Im x$. Now one has

$$x = (\Re x)_+ - (\Re x)_- + i\{(\Im x)_+ - (\Im x)_-\}. \quad (2.2)$$

Of course, $(\Re x)_\pm, (\Im x)_\pm \in \mathfrak{p}$. This is the meaning of (iv). \diamond

If $x - y \in \mathfrak{p}$, then we will write $x \geq y$ (or $y \leq x$) w.r.t. \mathfrak{p} . Let A and B be densely defined linear operators on \mathfrak{h} . If $Ax \geq Bx$ w.r.t. \mathfrak{p} for all $x \in \text{dom}(A) \cap \text{dom}(B) \cap \mathfrak{p}$, then we will write $A \supseteq B$ (or $B \preceq A$) w.r.t. \mathfrak{p} . Especially if A satisfies $0 \preceq A$ w.r.t. \mathfrak{p} , then we say that A *preserves positivity with respect to \mathfrak{p}* . The symbol “ \supseteq ” was first introduced by Miura [27].

An element x in \mathfrak{p} is called to be *strictly positive* if $\langle x, y \rangle \geq 0$ for all $y \in \mathfrak{p} \setminus \{0\}$. We will write this as $x > 0$ w.r.t. \mathfrak{p} . Of course, an inequality $x > y$ w.r.t. \mathfrak{p} means $x - y$ is strictly positive w.r.t. \mathfrak{p} . If bounded operators A and B satisfy $Ax > Bx$ w.r.t. \mathfrak{p} for all $x \in \mathfrak{p} \setminus \{0\}$, then we will express this as $A \triangleright B$ (or $B \triangleleft A$) w.r.t. \mathfrak{p} . Clearly if $A \triangleright B$ w.r.t. \mathfrak{p} , then $A \supseteq B$ w.r.t. \mathfrak{p} . We say that A *improves positivity w.r.t. \mathfrak{p}* if $A \triangleright 0$ w.r.t. \mathfrak{p} .

2.2 Monotonically decreasing self-adjoint operators

Let \mathfrak{p} be a self-dual cone in the Hilbert space \mathfrak{h} . Let $\{H_\lambda\}_{\lambda \geq 0}$ be a family of self-adjoint operators on \mathfrak{h} . In this section we always assume the following.

- (A. 1) There exists a constant $M > -\infty$, independent of λ , such that $H_\lambda \geq M$ for all $\lambda \geq 0$.
- (A. 2) For all $\lambda \geq 0$ and $s \geq 0$, $e^{-sH_\lambda} \supseteq 0$ w.r.t. \mathfrak{p} .
- (A. 3) Each H_λ has the common domain \mathcal{D} , i.e., $\text{dom}(H_\lambda) = \mathcal{D}$ for all $\lambda \geq 0$.

Under these assumptions, we can show the existence of the limiting hamiltonian as follows.

Theorem 2.3 *We assume (A. 1), (A. 2) and (A. 3). Suppose that H_λ is monotonically decreasing in λ in the sense that*

$$\lambda_1 \leq \lambda_2 \implies H_{\lambda_1} \supseteq H_{\lambda_2} \quad \text{w.r.t. } \mathfrak{p}. \quad (2.3)$$

Then, there exists a unique self-adjoint operator H , bounded from below by M , with following properties:

- (i) H_λ converges to H in strong resolvent sense as $\lambda \rightarrow \infty$,
- (ii) For all $\lambda \geq 0$ and $s \geq 0$, $e^{-sH} \supseteq e^{-sH_\lambda}$ w.r.t. \mathfrak{p} . In particular, $e^{-sH} \supseteq 0$ w.r.t. \mathfrak{p} for all $s \geq 0$.

In the above theorem, we proved the limiting hamiltonian denoted by H satisfies $e^{-sH} \supseteq 0$ w.r.t. \mathfrak{p} for all $s \geq 0$. Our next problem is to show $e^{-sH} \triangleright 0$ w.r.t. \mathfrak{p} for all $s > 0$. This problem is important because if we can solve it, then by the Perron-Frobenius-Faris theorem(Theorem A.3), the ground state of H (if it exists) is automatically unique and strictly positive w.r.t. \mathfrak{p} . Below we will give two answers about this problem.

Theorem 2.4 *In addition to the assumptions in Theorem 2.3, assume that, there exists a $\lambda \geq 0$ such that*

$$e^{-sH_\lambda} \triangleright 0 \quad (2.4)$$

w.r.t. \mathfrak{p} for all $s > 0$. Let H be the self-adjoint operator obtained in Theorem 2.3. Then one has

$$e^{-sH} \triangleright 0 \quad (2.5)$$

w.r.t. \mathfrak{p} for all $s > 0$. Consequently, if $\inf \text{spec}(H)$ is an eigenvalue, then it is unique and the corresponding eigenvector can be chosen as strictly positive w.r.t. \mathfrak{p} .

Theorem 2.4 was applied to show the uniqueness of the ground state for the polaron and bipolaron Hamiltonians without cutoffs in [28, 29]. Instead, the following theorem is convenient for concrete applications in this paper.

Theorem 2.5 *In addition to the assumptions in Theorem 2.3, assume that, for some $\mu \geq 0$, a family of self-adjoint operators $\{(H_\lambda + \mu)^{-1}\}_{\lambda \geq 0}$ is ergodic in the sense that, for any $x, y \in \mathfrak{p} \setminus \{0\}$, there exists a $\lambda \geq 0$ so that $\langle x, (H_\lambda + \mu)^{-1}y \rangle > 0$. Let H be the self-adjoint operator in Theorem 2.3. Then one has*

$$e^{-sH} \triangleright 0 \quad (2.6)$$

w.r.t. \mathfrak{p} for all $s > 0$. Consequently, if $\inf \text{spec}(H)$ is an eigenvalue, then it is unique and the corresponding eigenvector can be chosen as strictly positive w.r.t. \mathfrak{p} .

The operator monotonicity (2.3) further gives useful information concerning spectrum of the Hamiltonians. Indeed, in our previous work [31], the following theorem was proven.

Theorem 2.6 [31] *Assume (A. 1), (A. 2), (A. 3), (2.3). Set $E_\lambda = \inf \text{spec}(H_\lambda)$ and $E = \inf \text{spec}(H)$. Then E_λ is monotonically decreasing in λ and $\lim_{\lambda \rightarrow \infty} E_\lambda = E$.*

3 Proofs of Theorems 2.3, 2.4, 2.5

3.1 Proof of Theorem 2.3

STEP 1. Let $x, y \in \mathfrak{p}$. Define a function $F_{x,y}(\lambda)$ by $F_{x,y}(\lambda) = \langle x, e^{-H_\lambda}y \rangle$. Then, by (2.3), the function $F_{x,y}(\lambda)$ is monotonically increasing in λ and, by the assumption (A. 1), it is bounded. Hence $\lim_{\lambda \rightarrow \infty} F_{x,y}(\lambda)$ exists. Then, since \mathfrak{h} is linearly spanned by \mathfrak{p} (Remark 2.2), the limit $\lim_{\lambda \rightarrow \infty} \langle x, e^{-sH_\lambda}y \rangle$ exists for all $x, y \in \mathfrak{h}$. Now one can define a sesquilinear form $B_s(\cdot, \cdot)$ by

$$B_s(x, y) = \lim_{\lambda \rightarrow \infty} \langle x, e^{-sH_\lambda}y \rangle. \quad (3.7)$$

By the assumption (A. 1), one sees $|B_s(x, y)| \leq \|x\| \|y\| e^{-sM}$. Thus, by the Riesz's representation theorem, there exists a unique positive operator T_s such that

$$B_s(x, y) = \langle x, T_s y \rangle. \quad (3.8)$$

Then, by definition, one has

$$\|T_s\| \leq e^{-sM}, \quad T_s = \text{w-} \lim_{\lambda \rightarrow \infty} e^{-sH_\lambda}, \quad (3.9)$$

where w-lim means the weak limit. Moreover, by the monotonicity (2.3), one has

$$e^{-sH_\lambda} \leq T_s \quad (3.10)$$

for all $\lambda \geq 0$. [Proof: By Proposition A.1, one has $e^{-sH_{\lambda_2}} \leq e^{-sH_{\lambda_1}}$ whenever $\lambda_2 \geq \lambda_1$. Then taking $\lambda_2 \rightarrow \infty$, one concludes (3.10). Remark that to apply Proposition A.1, we used the assumptions (A. 2) and (A. 3).]

STEP 2. In this step we will show

$$\text{s-} \lim_{\lambda \rightarrow \infty} e^{-sH_\lambda} = T_s, \quad (3.11)$$

where s-lim means the strong limit.

Since \mathfrak{h} is linearly spanned by \mathfrak{p} (Remark 2.2), it suffices to show that

$$\text{s-}\lim_{\lambda \rightarrow \infty} e^{-sH_\lambda} x = T_s x \quad (3.12)$$

for all $x \in \mathfrak{p}$. To this end, observe that

$$\|(T_s - e^{-sH_\lambda})x\|^2 = \langle x, (T_s^2 - e^{-sH_\lambda} T_s - T_s e^{-sH_\lambda} + e^{-sH_\lambda} e^{-sH_\lambda})x \rangle. \quad (3.13)$$

Note that, by (3.10), one has $e^{-sH_\lambda} e^{-sH_\lambda} \leq T_s e^{-sH_\lambda}$ which implies $\langle x, e^{-sH_\lambda} e^{-sH_\lambda} x \rangle \leq \langle x, T_s e^{-sH_\lambda} x \rangle$. Hence

$$\begin{aligned} \text{RHS of (3.13)} &\leq \langle x, (T_s^2 - e^{-sH_\lambda} T_s - T_s e^{-sH_\lambda} + T_s e^{-sH_\lambda})x \rangle \\ &= \langle x, (T_s - e^{-sH_\lambda})T_s x \rangle \\ &\rightarrow 0 \end{aligned} \quad (3.14)$$

as $\lambda \rightarrow \infty$ by (3.9). This proves (3.12).

STEP 3. Since $\{e^{-sH_\lambda}\}_{s \geq 0}$ is a one-parameter semigroup, $\{T_s\}_{s \geq 0}$ is also one-parameter semigroup by STEP 2. If one can show the strong continuity of T_s in $s \geq 0$, one sees that there exists a unique self-adjoint operator H such that $T_s = e^{-sH}$ by Proposition B.2. By our construction, it is clear that

$$e^{-sH} = \text{s-}\lim_{\lambda \rightarrow \infty} e^{-sH_\lambda}, \quad e^{-sH} \geq e^{-sH_\lambda}, \quad H \geq M. \quad (3.15)$$

This is the assertion in the theorem.

Let prove the strong continuity of T_s in s . Without loss of generality, we may assume $M > 0$, i.e., $H_\lambda \geq M > 0$. (Indeed, if $M > 0$, there is nothing to do. On the other hand, if $M \leq 0$, then we simply study $\tilde{H}_\lambda = H_\lambda - M + \varepsilon$ ($\varepsilon > 0$) instead of H_λ itself. Obviously $\tilde{H}_\lambda \geq \varepsilon$ for all λ .) Hence $\mathbb{1} - T_s \geq 0$ holds for all $s \geq 0$. Fix $\lambda \geq 0$ arbitrarily. One has, by (3.10),

$$e^{-sH_\lambda} \leq T_s \quad (3.16)$$

w.r.t. \mathfrak{p} . Thus $\mathbb{1} - e^{-sH_\lambda} \geq \mathbb{1} - T_s$ w.r.t. \mathfrak{p} . Hence, for all $x \in \mathfrak{p}$, we have

$$0 \leq \langle x, (\mathbb{1} - T_s)x \rangle \leq \langle x, (\mathbb{1} - e^{-sH_\lambda})x \rangle \quad (3.17)$$

for all $s \geq 0$. Taking $s \rightarrow +0$,

$$\lim_{s \rightarrow +0} \langle x, (\mathbb{1} - T_s)x \rangle = 0 \quad (3.18)$$

holds for all $x \in \mathfrak{p}$. Next observe that, for any $x \in \mathfrak{p}$,

$$\|(\mathbb{1} - T_s)x\|^2 = \langle x, (\mathbb{1} - 2T_s + T_{2s})x \rangle \rightarrow 0 \quad (3.19)$$

as $s \rightarrow +0$ by (3.18). Then since \mathfrak{h} is linearly spanned by \mathfrak{p} (Remark 2.2), one sees $\|(T_s - \mathbb{1})x\| \rightarrow 0$ as $s \rightarrow +0$ for all $x \in \mathfrak{h}$. \square

3.2 Proof of Theorem 2.4

The proof of this theorem was already given in [29]. Here we will repeat it for reader's convenience.

By Theorem 2.3 (ii),

$$e^{-sH} \supseteq e^{-sH_\lambda} \quad (3.20)$$

w.r.t. \mathfrak{p} . Since $e^{-sH_\lambda} \triangleright 0$, one arrives at

$$e^{-sH} \supseteq e^{-sH_\lambda} \triangleright 0 \quad (3.21)$$

w.r.t. \mathfrak{p} for all $s > 0$. Remainder assertions comes from Theorem A.3. \square

3.3 Proof of Theorem 2.5

By (2.3) and Proposition A.1, one has

$$(H + \mu)^{-1} \supseteq (H_\lambda + \mu)^{-1} \quad (3.22)$$

w.r.t. \mathfrak{p} for all $\lambda \geq 0$. Choose $x, y \in \mathfrak{p} \setminus \{0\}$ arbitrarily. Then, by the ergodicity of $\{(H_\lambda + \mu)^{-1}\}_{\lambda \geq 0}$, there is a $\lambda \geq 0$ such that $\langle x, (H_\lambda + \mu)^{-1} y \rangle > 0$. Combining this with (3.22), we have

$$\langle x, (H + \mu)^{-1} y \rangle \geq \langle x, (H_\lambda + \mu)^{-1} y \rangle > 0. \quad (3.23)$$

This means $(H + \mu)^{-1} \triangleright 0$ w.r.t. \mathfrak{p} . Now we apply Theorem A.3 and conclude the assertion. \square

4 Second quantization

Before we study the Fröhlich polaron, we briefly summarize necessary results of the second quantization. Many of these have already been stated in the previous work [31].

4.1 Basic definitions

The bosonic Fock space over \mathfrak{h} is defined by

$$\mathfrak{F}(\mathfrak{h}) = \sum_{n \geq 0}^{\oplus} \mathfrak{F}^{(n)}(\mathfrak{h}), \quad \mathfrak{F}^{(n)}(\mathfrak{h}) = \mathfrak{h}^{\otimes_{\text{s}} n}, \quad (4.1)$$

where $\mathfrak{h}^{\otimes_{\text{s}} n}$ is the n -fold symmetric tensor product of \mathfrak{h} with convention $\mathfrak{h}^{\otimes_{\text{s}} 0} = \mathbb{C}$. $\mathfrak{F}^{(n)}(\mathfrak{h})$ is called the n -boson subspace.

We denote by $a(f)$ ($f \in \mathfrak{h}$) the annihilation operator on $\mathfrak{F}(\mathfrak{h})$, its adjoint $a(f)^*$, called the creation operator, is defined by

$$a(f)^* \varphi = \sum_{n \geq 1}^{\oplus} \sqrt{n} S_n(f \otimes \varphi^{(n-1)}) \quad (4.2)$$

for $\varphi = \sum_{n \geq 0}^{\oplus} \varphi^{(n)} \in \text{dom}(a(f)^*)$, where S_n is the symmetrizer on $\mathfrak{F}^{(n)}(\mathfrak{h}) = \mathfrak{h}^{\otimes_{\text{s}} n}$. The annihilation- and creation operators satisfy the canonical commutation relations (CCRs)

$$[a(f), a(g)^*] = \langle f, g \rangle, \quad [a(f), a(g)] = 0 = [a(f)^*, a(g)^*] \quad (4.3)$$

on a suitable dense subspace in $\mathfrak{F}(\mathfrak{h})$.

Let C be a contraction operator on \mathfrak{h} , that is, $\|C\| \leq 1$. Then we define a contraction operator $\Gamma(C)$ on $\mathfrak{F}(\mathfrak{h})$ by

$$\Gamma(C) = \sum_{n \geq 0}^{\oplus} C^{\otimes n} \quad (4.4)$$

with $C^{\otimes 0} = \mathbb{1}$, the identity operator. For a self-adjoint operator A on \mathfrak{h} , let us introduce

$$d\Gamma(A) = 0 \oplus \sum_{n \geq 1}^{\oplus} \sum_{n \geq k \geq 1} \mathbb{1}^{\otimes(k-1)} \otimes A \otimes \mathbb{1}^{\otimes(n-k)} \quad (4.5)$$

acting in $\mathfrak{F}(\mathfrak{h})$. Then $d\Gamma(A)$ is essentially self-adjoint. We denote its closure by the same symbol. A typical example is the bosonic number operator $N_f = d\Gamma(\mathbb{1})$. We remark the following relation between $\Gamma(\cdot)$ and $d\Gamma(\cdot)$:

$$\Gamma(e^{itA}) = e^{itd\Gamma(A)}. \quad (4.6)$$

In particular if A is positive, then one has

$$\Gamma(e^{-tA}) = e^{-td\Gamma(A)}. \quad (4.7)$$

Proposition 4.1 *Let A be a positive self-adjoint operator. Then we have the following operator inequalities:*

$$a(f)^* a(f) \leq \|A^{-1/2} f\|^2 (d\Gamma(A) + \mathbb{1}), \quad (4.8)$$

$$d\Gamma(A) + a(f) + a(f)^* \geq -\|A^{-1/2} f\|^2. \quad (4.9)$$

4.2 Fock space over L^2 -space

In this paper, the bosonic Fock space over $L^2(\mathbb{R}_k^3) = L^2(\mathbb{R}^3, dk)$ will often appear and we simply denote as

$$\mathfrak{F} = \mathfrak{F}(L^2(\mathbb{R}_k^3)). \quad (4.10)$$

The n -boson subspace $\mathfrak{F}^{(n)} = L^2(\mathbb{R}_k^3)^{\otimes n}$ is naturally identified with $L_{\text{sym}}^2(\mathbb{R}^{3n}) = \{\varphi \in L^2(\mathbb{R}_k^{3n}) \mid \varphi(k_1, \dots, k_n) = \varphi(k_{\sigma(1)}, \dots, k_{\sigma(n)}) \text{ a.e. } \forall \sigma \in \mathfrak{S}_n\}$, where \mathfrak{S}_n is the permutation group on a set $\{1, 2, \dots, n\}$. Hence

$$\mathfrak{F} = \mathbb{C} \oplus \sum_{n \geq 1}^{\oplus} L_{\text{sym}}^2(\mathbb{R}_k^{3n}). \quad (4.11)$$

The annihilation- and creation operators are symbolically expressed as

$$a(f) = \int_{\mathbb{R}^3} dk \overline{f(k)} a(k), \quad a(f)^* = \int_{\mathbb{R}^3} dk f(k) a(k)^*. \quad (4.12)$$

If ω is a multiplication operator by the function $\omega(k)$, then $d\Gamma(\omega)$ is formally written as

$$d\Gamma(\omega) = \int_{\mathbb{R}_k^3} dk \omega(k) a(k)^* a(k). \quad (4.13)$$

4.3 The Fröhlich cone

In order to discuss the inequalities introduced in §2, we have to determine a self-dual cone in \mathfrak{F} . Here we will introduce a natural self-dual cone in \mathfrak{F} which is suitable for our analysis in later sections.

Under the natural identification $L^2(\mathbb{R}_k^3)^{\otimes n} = L_{\text{sym}}^2(\mathbb{R}_k^{3n})$, set

$$\mathfrak{F}_+^{(n)} = \{\varphi \in L_{\text{sym}}^2(\mathbb{R}_k^{3n}) \mid \varphi(k_1, \dots, k_n) \geq 0 \text{ a.e.}\} \quad (4.14)$$

with $\mathfrak{F}_+^{(0)} = \mathbb{R}_+ = \{r \in \mathbb{R} \mid r \geq 0\}$. Then each $\mathfrak{F}_+^{(n)}$ is a self-dual cone in $\mathfrak{F}^{(n)} = L_{\text{sym}}^2(\mathbb{R}_k^{3n})$.

Definition 4.2 The *Fröhlich cone* is defined by

$$\mathfrak{F}_+ = \sum_{n \geq 0}^{\oplus} \mathfrak{F}_+^{(n)}. \quad (4.15)$$

Again \mathfrak{F}_+ is a self-dual cone in \mathfrak{F} . \diamond

We summarize properties of operators in \mathfrak{F} below. All propositions were proven in [31].

Proposition 4.3 Let C be a contraction on $L^2(\mathbb{R}_k^3)$. Then if $C \geq 0$ w.r.t. $L^2(\mathbb{R}_k^3)_+$, one has $\Gamma(C) \geq 0$ w.r.t. \mathfrak{F}_+ , where $L^2(\mathbb{R}_k^3) = \{f \in L^2(\mathbb{R}_k^3) \mid f(k) \geq 0 \text{ a.e.}\}$. Especially one has the following.

- (i) For a self-adjoint operator A , if $e^{itA} \geq 0$ w.r.t. $L^2(\mathbb{R}_k^3)_+$, then one has $\Gamma(e^{itA}) \geq 0$ w.r.t. \mathfrak{F}_+ .
- (ii) For a positive self-adjoint operator B , if $e^{-tB} \geq 0$ w.r.t. $L^2(\mathbb{R}_k^3)_+$, then one has $\Gamma(e^{-tB}) \geq 0$ w.r.t. \mathfrak{F}_+ .

Proposition 4.4 If $f \geq 0$ w.r.t. $L^2(\mathbb{R}_k^3)_+$, then $a(f)^* \geq 0$ and $a(f) \geq 0$ w.r.t. \mathfrak{F}_+ .

Proposition 4.5 (Ergodicity) For each $f \in L^2(\mathbb{R}_k^3)$, let $\phi(f)$ be a linear operator defined by

$$\phi(f) = a(f) + a(f)^*. \quad (4.16)$$

If $f > 0$ w.r.t. $L^2(\mathbb{R}_k^3)_+$, that is, $f(k) > 0$ a.e. k , then $\phi(f)$ is ergodic in the sense that, for any $x, y \in (\mathfrak{F}_+ \cap \mathfrak{F}_{\text{fin}}) \setminus \{0\}$, there exists an $n \in \{0\} \cup \mathbb{N}$ such that $\langle x, \phi(f)^n y \rangle > 0$.

4.4 Local properties

Let B_Λ be a ball of radius Λ in \mathbb{R}_k^3 and let χ_Λ be a function on \mathbb{R}^3 defined by $\chi_\Lambda(k) = 1$ if $k \in B_\Lambda$ and $\chi_\Lambda(k) = 0$ otherwise. Then as a multiplication operator, χ_Λ is an orthogonal projection on $L^2(\mathbb{R}_k^3)$ and $Q_\Lambda = \Gamma(\chi_\Lambda)$ is also an orthogonal projection on \mathfrak{F} . Now let us define the local Fock space by

$$\mathfrak{F}_\Lambda = Q_\Lambda \mathfrak{F}. \quad (4.17)$$

Clearly $\mathfrak{F} = \mathfrak{F}_\infty$. Since $\chi_\Lambda L^2(\mathbb{R}_k^3) = L^2(B_\Lambda)$, \mathfrak{F}_Λ can be identified with $\mathfrak{F}(L^2(B_\Lambda))$.

The following proposition was unstated in [31].

Proposition 4.6 For each $\Lambda \geq 0$, put $Q_\Lambda^\perp = \mathbb{1} - Q_\Lambda$. Then one obtains the following.

(i) $Q_\Lambda \geq 0$ w.r.t. \mathfrak{F}_+ .

(ii) $Q_\Lambda^\perp \geq 0$ w.r.t. \mathfrak{F}_+ .

Proof. (i) immediately follows from Proposition 4.3.

(ii) Under the identification (4.11), we see

$$(Q_\Lambda \varphi_n)(k_1, \dots, k_n) = \prod_{j=1}^n \chi_\Lambda(k_j) \varphi_n(k_1, \dots, k_n) \quad (4.18)$$

for each $\varphi_n \in L_{\text{sym}}^2(\mathbb{R}_k^{3n})$. Hence

$$(Q_\Lambda^\perp \varphi_n)(k_1, \dots, k_n) = \left\{ 1 - \prod_{j=1}^n \chi_\Lambda(k_j) \right\} \varphi_n(k_1, \dots, k_n). \quad (4.19)$$

If $\varphi_n(k_1, \dots, k_n) \geq 0$ a.e., then the right hand side of (4.19) is positive for a.e. k_1, \dots, k_n because $1 - \prod_{j=1}^n \chi_\Lambda(k_j) \geq 0$. This means $Q_\Lambda^\perp \geq 0$ w.r.t. \mathfrak{F}_+ . \square

As to the annihilation- and creation operators, we remark the following properties:

$$a(f)Q_\Lambda = a(\chi_\Lambda f) = \int_{|k| \leq \Lambda} dk \overline{f(k)} a(k), \quad (4.20)$$

$$Q_\Lambda a(f)^* = a(\chi_\Lambda f)^* = \int_{|k| \leq \Lambda} dk f(k) a(k)^*, \quad (4.21)$$

$$d\Gamma(\omega)Q_\Lambda = d\Gamma(\chi_\Lambda \omega) = \int_{|k| \leq \Lambda} dk \omega(k) a(k)^* a(k). \quad (4.22)$$

Next let us introduce a natural self-dual cone in \mathfrak{F}_Λ . To this end, define

$$\mathfrak{F}_{\Lambda,+}^{(n)} = \{ \varphi \in L_{\text{sym}}^2(B_\Lambda^{\times n}) \mid \varphi(k_1, \dots, k_n) \geq 0 \text{ a.e.} \} \quad (4.23)$$

with $\mathfrak{F}_{\Lambda,+}^{(0)} = \mathbb{R}^+$. Each $\mathfrak{F}_{\Lambda,+}^{(n)}$ is a self-dual cone in $L^2(B_\Lambda)^{\otimes n} = L_{\text{sym}}^2(B_\Lambda^{\times n})$.

Definition 4.7 The *local Fröhlich cone* is defined by

$$\mathfrak{F}_{\Lambda,+} = \sum_{n \geq 0}^{\oplus} \mathfrak{F}_{\Lambda,+}^{(n)}. \quad (4.24)$$

$\mathfrak{F}_{\Lambda,+}$ is a self-dual cone in \mathfrak{F}_Λ . \diamond

Proposition 4.8 Propositions 4.3, 4.4 and 4.5 are still true even if one replaces $L^2(\mathbb{R}_k^3)_+$ and \mathfrak{F}_+ by $L^2(B_\Lambda)_+$ and $\mathfrak{F}_{\Lambda,+}$ respectively.

As to the proof of the above proposition, see [31].

5 The polaron: From a viewpoint of operator inequalities

5.1 Definition of Hamiltonians with an ultraviolet cutoff

The Fröhlich polaron model describes an electron in an ionic crystal [14]. Despite its long history, this topic is still being studied actively [7, 12, 13, 19, 20, 24, 31]. The literature on this model is vast and we content ourselves with mentioning two references [5, 11]. Although the structure of the model is simple, the model has rich contents both mathematically and physically. For this reason, many researchers employ the model as a touchstone, by applying their own methods [1, 2, 6, 10, 21, 23, 28, 36, 39, 40]. As a test of our theory of the operator monotonicity, we will choose this model as well.

5.1.1 The Fröhlich Hamiltonian

For each $\Lambda > 0$, we define the *Fröhlich Hamiltonian* with an ultraviolet cutoff Λ by

$$H_\Lambda = -\frac{1}{2}\Delta_x - \sqrt{\alpha}\lambda_0 \int_{|k|\leq\Lambda} dk \frac{1}{|k|} [e^{ik\cdot x} a(k) + e^{-ik\cdot x} a(k)] + N_f, \quad (5.1)$$

where Δ_x is the Laplacian on $L^2(\mathbb{R}_x^3)$ and $\lambda_0 = 2^{1/4}(2\pi)^{-1}$. H_Λ acts in the Hilbert space $L^2(\mathbb{R}_x^3) \otimes \mathfrak{F}$. Then, by the standard bound

$$\|a(f)^\#(N_f + \mathbb{1})^{-1/2}\| \leq \|f\| \quad (5.2)$$

which comes from (4.8), and Kato-Rellich theorem [37], H_Λ is self-adjoint on $\text{dom}(\Delta_x) \cap \text{dom}(N_f)$ and bounded from below.

5.1.2 The Fröhlich Hamiltonian at a fixed total momentum

Let P_{tot} be the total momentum operator defined by

$$P_{\text{tot}} = -i\nabla_x + P_f. \quad (5.3)$$

Let \mathcal{F}_x be the Fourier transformation on $L^2(\mathbb{R}_x^3)$ and let $U = \mathcal{F}_x e^{ix\cdot P_f}$. This unitary operator U gives a spectral representation of P_{tot} , namely,

$$UP_{\text{tot}}U^* = \int_{\mathbb{R}^3}^{\oplus} P dP. \quad (5.4)$$

Moreover one has

$$UH_\Lambda U^* = \int_{\mathbb{R}^3}^{\oplus} H_\Lambda(P) dP. \quad (5.5)$$

Each $H_\Lambda(P)$ is concretely expressed as

$$H_\Lambda(P) = \frac{1}{2}(P - P_f)^2 - \sqrt{\alpha}\lambda_0 \int_{|k|\leq\Lambda} dk \frac{1}{|k|} [a(k) + a(k)^*] + N_f. \quad (5.6)$$

Then, by (5.2) and Kato-Rellich theorem, $H_\Lambda(P)$ is self-adjoint on $\text{dom}(P_f^2) \cap \text{dom}(N_f)$, bounded from below. The self-adjoint operator (5.6) is called the Fröhlich Hamiltonian with an ultraviolet cutoff Λ , at a fixed total momentum P .

5.2 Results

5.2.1 The Fröhlich Hamiltonian

To investigate properties of the total Hamiltonian H_Λ , it is convenient to move to the electron momentum space: $L^2(\mathbb{R}_p^3) \otimes \mathfrak{F} = \mathcal{F}_x L^2(\mathbb{R}_x^3) \otimes \mathfrak{F}$. In order to define the inequalities \supseteq etc., we have to fix a suitable self-dual cone in $L^2(\mathbb{R}_p^3) \otimes \mathfrak{F}$. To this end, let

$$\mathfrak{P} = \{ \Psi \in L^2(\mathbb{R}_p^3) \otimes \mathfrak{F} \mid \forall f \in L^2(\mathbb{R}_p^3)_+ \ \forall \varphi \in \mathfrak{F}_+, \langle \Psi, f \otimes \varphi \rangle \geq 0 \}. \quad (5.7)$$

Then one sees that \mathfrak{P} is a self-dual cone. Moreover \mathfrak{P} can be represented as

$$\mathfrak{P} = \sum_{n \geq 0}^{\oplus} \mathfrak{P}_n \quad (5.8)$$

with $\mathfrak{P}_0 = L^2(\mathbb{R}_p^3)_+$ and $\mathfrak{P}_n = \{ \Psi \in L^2(\mathbb{R}_p^3) \otimes L_{\text{sym}}^2(\mathbb{R}_k^{3n}) \mid \langle \Psi, f \otimes \varphi \rangle > 0 \ \forall f \in L^2(\mathbb{R}_p^3)_+ \forall \varphi \in \mathfrak{F}_+^{(n)} \}$. Remark that, under the identification (4.14), one sees

$$\mathfrak{P}_n = \{ \Psi \in L^2(\mathbb{R}_p^3) \otimes L_{\text{sym}}^2(\mathbb{R}_k^{3n}) \mid \Psi(p; k_1, \dots, k_n) \geq 0 \text{ a.e.} \}. \quad (5.9)$$

Now let us display our results. Our first result is as follows.

Theorem 5.1 *One has the following.*

- (i) *For any $\Lambda > 0$, there exists a constant M independent of Λ such that $H_\Lambda \geq M$.*
- (ii) *For any $\Lambda > 0$, $\mathcal{F}_x e^{-tH_\Lambda} \mathcal{F}_x^{-1} \supseteq 0$ w.r.t. \mathfrak{P} for all $t \geq 0$, where \mathcal{F}_x is the Fourier transformation associated with x .*
- (iii) *$\mathcal{F}_x H_\Lambda \mathcal{F}_x^{-1}$ is monotonically decreasing in Λ in the sense*

$$\Lambda' \geq \Lambda \implies \mathcal{F}_x H_\Lambda \mathcal{F}_x^{-1} \supseteq \mathcal{F}_x H_{\Lambda'} \mathcal{F}_x^{-1} \quad \text{w.r.t. } \mathfrak{P}. \quad (5.10)$$

A proof of Theorem 5.1 will be given in §6. Now we are ready to apply Theorem 2.3. An immediate corollary is as follows.

Corollary 5.2 *There exists a self-adjoint operator H , bounded from below by M , with the following properties.*

- (i) *H_Λ converges to H in strong resolvent sense as $\Lambda \rightarrow \infty$.*
- (ii) *For all $\Lambda \geq 0$ and $s \geq 0$, $\mathcal{F}_x e^{-sH} \mathcal{F}_x^{-1} \supseteq \mathcal{F}_x e^{-sH_\Lambda} \mathcal{F}_x^{-1}$ w.r.t. \mathfrak{P} . In particular, $\mathcal{F}_x e^{-sH} \mathcal{F}_x^{-1} \supseteq 0$ w.r.t. \mathfrak{P} for all $s \geq 0$.*

Remark 5.3 There are several ways to define the Hamiltonian H as a limiting operator [18, 21, 35, 39]. Here we propose a novel method by the operator monotonicity. To keep a decent form of the article, we exhibit results here, however the readers should pay attention to our proof. \diamond

We will give a proof of Corollary 5.2 in §7. By the above corollary, we can define the Fröhlich Hamiltonian without ultraviolet cutoff by H .

5.2.2 The Fröhlich Hamiltonian at a fixed total momentum

Next we will state results on $H_\Lambda(P)$.

Theorem 5.4 *One has the following.*

- (i) *For any $\Lambda > 0$, there exists a constant M independent of Λ and P such that $H_\Lambda(P) \geq M$.*
- (ii) *For any $\Lambda > 0$, $e^{-tH_\Lambda(P)} \geq 0$ w.r.t. \mathfrak{F}_+ for all $t \geq 0$ and $P \in \mathbb{R}^3$.*
- (iii) *For all $P \in \mathbb{R}^3$, $H_\Lambda(P)$ is monotonically decreasing in Λ in the sense*

$$\Lambda' \geq \Lambda \implies H_\Lambda(P) \geq H_{\Lambda'}(P) \quad \text{w.r.t. } \mathfrak{F}_+. \quad (5.11)$$

We will show Theorem 5.4 in §8. Applying Theorem 2.3, one has the following corollary.

Corollary 5.5 *There exists a self-adjoint operator $H(P)$, bounded from below by M , with the following properties.*

- (i) *$H_\Lambda(P)$ converges to $H(P)$ in strong resolvent sense as $\Lambda \rightarrow \infty$.*
- (ii) *For all $\Lambda \geq 0$, $P \in \mathbb{R}^3$ and $s \geq 0$, $e^{-sH(P)} \geq e^{-sH_\Lambda(P)}$ w.r.t. \mathfrak{F}_+ . In particular, $e^{-sH(P)} \geq 0$ w.r.t. \mathfrak{F}_+ for all $P \in \mathbb{R}^3$ and $s \geq 0$.*
- (iii) *Let H be the Hamiltonian in Theorem 5.2. Then one has*

$$UHU^{-1} = \int_{\mathbb{R}^3}^{\oplus} H(P) \, dP. \quad (5.12)$$

We will give a proof of Corollary 5.5 in §9. In this way, we can define the Hamiltonian without ultraviolet cutoff by $H(P)$.

As to the limiting Hamiltonian $H(P)$, we can say more.

Theorem 5.6 *Let $H(P)$ be the self-adjoint operator in Theorem 5.5. Then one has*

$$e^{-sH(P)} \triangleright 0 \quad (5.13)$$

w.r.t. \mathfrak{F}_+ for all $P \in \mathbb{R}^3$ and $s > 0$. Consequently, if $\inf \text{spec}(H(P))$ is an eigenvalue, then it is unique and the corresponding eigenvector can be chosen as strictly positive w.r.t. \mathfrak{F}_+ .

We will prove Theorem 5.6 in §10. Existence of a ground state of $H(P)$ was fully understood, namely, it was already shown $H(P)$ has a ground state provided $|P| < \sqrt{2}$ [17, 41]. Combining this and Theorem A.3, we arrive at the corollary below.

Corollary 5.7 *For all $P \in \mathbb{R}^3$ with $|P| < \sqrt{2}$, $H(P)$ has a unique ground state which is strictly positive w.r.t. \mathfrak{F}_+ .*

5.2.3 Monotonicity of the polaron energy [31]

By the operator monotonicity (5.11), we can further obtain information about the ground state energy of $H_\Lambda(P)$. Here we only exhibit the results proved in [31].

Theorem 5.8 [31] *Under assumptions in Theorem 5.5, set $E_\Lambda(P) = \inf \text{spec}(H_\Lambda(P))$ and $E(P) = \inf \text{spec}(H(P))$. Then, for all $|P| < \sqrt{2}$, $E_\Lambda(P)$ is strictly decreasing in Λ .*

Let $E_\Lambda = \inf \text{spec}(H_\Lambda)$. Then, by (5.5) and the fact $E_\Lambda(0) \leq E_\Lambda(P)$, one has $E_\Lambda = E_\Lambda(0)$. This equality implies the following.

Corollary 5.9 *Under assumptions in Theorem 5.5, set $E = \inf \text{spec}(H)$. Then E_Λ is strictly decreasing in Λ .*

In this way, our method of the operator monotonicity gives a consistent theory of the Fröhlich polaron.

6 Proof of Theorem 5.1

6.1 Proof of (i): Uniform lower bound

Proposition 6.1 *Take $0 < \Lambda_0 < \infty$ such that*

$$\alpha\lambda_0^2 \int_{|k| \geq \Lambda_0} dk \frac{1}{|k|^4} < \frac{1}{8}. \quad (6.1)$$

Then, for any Λ satisfying $\Lambda \geq \Lambda_0$, one has

$$H_\Lambda \geq -\alpha\lambda_0^2 \int_{|k| \leq \Lambda_0} dk \frac{1}{|k|^2} - \frac{1}{2}.$$

Proof. We will employ a method established by Lieb-Yamazaki [26]. (See also [25].) Pick Λ such that $\Lambda \geq \Lambda_0$. Let $\chi_\Lambda(k) = 1$ if $|k| \leq \Lambda$, $\chi_\Lambda(k) = 0$ otherwise. Define $\mathbf{Z} = (Z_1, Z_2, Z_3)$ by

$$Z_j = \sqrt{\alpha}\lambda_0 \int_{\mathbb{R}_k^3} dk k_j \frac{\chi_\Lambda(k) - \chi_{\Lambda_0}(k)}{|k|^3} e^{ik \cdot x} a(k). \quad (6.2)$$

Note that $k_j(\chi_\Lambda(k) - \chi_{\Lambda_0}(k))/|k|^3$ is square-integrable. Set

$$I(f) = \sqrt{\alpha}\lambda_0 \int_{\mathbb{R}_k^3} dk \frac{f(k)}{|k|} [e^{ik \cdot x} a(k) + e^{-ik \cdot x} a(k)^*]. \quad (6.3)$$

Then, using basic operator inequalities $2^{-1}\varepsilon a^*a + 2\varepsilon^{-1}b^*b \geq \pm(a^*b + b^*a)$ for any $\varepsilon > 0$, one sees

$$\begin{aligned}
I(\chi_\Lambda - \chi_{\Lambda_0}) &= \sum_{j=1,2,3} [-i\nabla_x, Z_j - Z_j^*] \\
&\leq \varepsilon(-\Delta_x) + 2\varepsilon^{-1}(\mathbf{Z}^* \cdot \mathbf{Z} + \mathbf{Z} \cdot \mathbf{Z}^*) \\
&= \varepsilon(-\Delta_x) + 4\varepsilon^{-1}\mathbf{Z}^* \cdot \mathbf{Z} + 2\varepsilon^{-1}\alpha\lambda_0^2 \int dk \frac{(\chi_\Lambda(k) - \chi_{\Lambda_0}(k))^2}{|k|^4} \quad (\text{By CCRs}) \\
&\leq \varepsilon(-\Delta_x) + 4\varepsilon^{-1}\alpha\lambda_0^2 \int dk \frac{\chi_\Lambda(k) - \chi_{\Lambda_0}(k)}{|k|^4} d\Gamma(\chi_\Lambda - \chi_{\Lambda_0}) \\
&\quad + 2\varepsilon^{-1}\alpha\lambda_0^2 \int dk \frac{\chi_\Lambda(k) - \chi_{\Lambda_0}(k)}{|k|^4}. \quad (\text{By (4.8)})
\end{aligned} \tag{6.4}$$

Take $\varepsilon = 4\alpha\lambda_0^2 \int dk (\chi_\Lambda(k) - \chi_{\Lambda_0}(k))/|k|^4$. Then (6.4) becomes

$$I(\chi_\Lambda - \chi_{\Lambda_0}) \leq \varepsilon(-\Delta_x) + d\Gamma(\mathbb{1} - \chi_{\Lambda_0}) + \frac{1}{2}. \tag{6.5}$$

Hence, by (4.9),

$$\begin{aligned}
H_\Lambda &= H_{\Lambda_0} - I(\chi_\Lambda - \chi_{\Lambda_0}) \\
&\geq \frac{1}{2}(1 - 2\varepsilon)(-\Delta_x) - I(\chi_{\Lambda_0}) + d\Gamma(\chi_{\Lambda_0}) - \frac{1}{2} \\
&\geq d\Gamma(\chi_{\Lambda_0}) - I(\chi_{\Lambda_0}) - \frac{1}{2} \\
&\geq -\alpha\lambda_0^2 \int dk \frac{\chi_{\Lambda_0}(k)}{|k|^2} - \frac{1}{2}.
\end{aligned} \tag{6.6}$$

This proves the assertion. \square

Fix Λ_0 arbitrarily so that (6.1) holds. Then, for each $\Lambda \geq \Lambda_0$, we have

$$H_\Lambda \geq -\alpha\lambda_0^2 \int_{|k| \leq \Lambda_0} dk \frac{1}{|k|} - \frac{1}{2} \tag{6.7}$$

by Proposition 6.1. On the other hand, for $0 \leq \Lambda \leq \Lambda_0$, one sees

$$\begin{aligned}
H_\Lambda &\geq N_f - \sqrt{\alpha}\lambda_0 \int_{|k| \leq \Lambda} dk \frac{1}{|k|} [e^{ik \cdot x} a(k) + e^{-ik \cdot x} a(k)^*] \\
&\geq -\alpha\lambda_0^2 \int_{|k| \leq \Lambda} dk \frac{1}{|k|^2}
\end{aligned} \tag{6.8}$$

by (4.9). Combining (6.7) with (6.8), one arrives at

$$H_\Lambda \geq -\alpha\lambda_0^2 \int_{|k| \leq \Lambda_0} dk \frac{1}{|k|} - \frac{1}{2} \tag{6.9}$$

for any $\Lambda \geq 0$. \square

6.2 Proof of (ii): Positivity preserving property

Let us denote a transformed Hamiltonian $\mathcal{F}_x H_\Lambda \mathcal{F}_x^{-1}$ by \hat{H}_Λ . \hat{H}_Λ acts in the Hilbert space $L^2(\mathbb{R}_p^3) \otimes \mathfrak{F} = \mathcal{F}_x L^2(\mathbb{R}_x^3) \otimes \mathfrak{F}$. In addition, \hat{H}_Λ has the following form

$$\hat{H}_\Lambda = \mathcal{F}_x H_\Lambda \mathcal{F}_x^{-1} = \hat{H}_0 - \hat{V}_\Lambda \quad (6.10)$$

with

$$\hat{H}_0 = \frac{1}{2}p^2 + N_f, \quad (6.11)$$

$$\hat{V}_\Lambda = \sqrt{\alpha}\lambda_0 \int_{|k| \leq \Lambda} dk \frac{1}{|k|} [e^{ik \cdot (-i\nabla_p)} a(k) + e^{-ik \cdot (-i\nabla_p)} a(k)^*], \quad (6.12)$$

where p^2 is a multiplication operator and ∇_p is the standard nabla symbol on $L^2(\mathbb{R}_p^3)$. As we will see, the expression (6.12) is essential for our proof.

Lemma 6.2 (Attraction) *For any $\Lambda \geq 0$, $-V_\Lambda$ is attractive in the sense $-V_\Lambda \trianglelefteq 0$ w.r.t. \mathfrak{P} .*

Proof. Recall the expression (6.12). Since $e^{ik \cdot (-i\nabla_p)}$ is a translation, it satisfies $e^{ik \cdot (-i\nabla_p)} \trianglelefteq 0$ w.r.t. $L^2(\mathbb{R}_p^3)_+$. Hence one also concludes, for any $\Lambda \geq 0$,

$$\int_{|k| \leq \Lambda} dk \frac{1}{|k|} e^{ik \cdot (-i\nabla_p)} a(k) \trianglelefteq 0 \quad (6.13)$$

w.r.t. \mathfrak{P} . Accordingly its adjoint operator is also positivity preserving. Now we conclude the assertion in the lemma. \square

Since $e^{-s\mathbb{1}} \trianglelefteq 0$ w.r.t. $L^2(\mathbb{R}_k^3)_+$, one has, by Proposition 4.3, $e^{-sN_f} = \Gamma(e^{-s\mathbb{1}}) \trianglelefteq 0$ w.r.t. \mathfrak{F}_+ which implies $e^{-s(\frac{1}{2}p^2 + N_f)} = e^{-s\frac{1}{2}p^2} e^{-sN_f} \trianglelefteq 0$ w.r.t. \mathfrak{P} . Here we used the fact the multiplication operator $e^{-s\frac{1}{2}p^2}$ preserves the positivity w.r.t. \mathfrak{P} . Then we can apply Corollary A.2 and conclude $e^{-sH_\Lambda} \trianglelefteq 0$ w.r.t. \mathfrak{P} . This completes the proof of (ii). \square

6.3 Proof of (iii): Operator monotonicity

Let us begin with the following lemma.

Lemma 6.3 $-\hat{V}_\Lambda$ is monotonically decreasing in Λ in a sense that

$$\Lambda_1 \leq \Lambda_2 \implies -\hat{V}_{\Lambda_1} \trianglelefteq -\hat{V}_{\Lambda_2} \quad \text{w.r.t. } \mathfrak{P}. \quad (6.14)$$

Remark 6.4 The attraction becomes stronger, the larger we take the ultraviolet cutoff. This is the physical meaning of the above lemma.

Proof. Suppose $\Lambda_1 \leq \Lambda_2$. Then, for each $\varphi \in \text{dom}(\hat{V}_{\Lambda_1}) \cap \text{dom}(\hat{V}_{\Lambda_2})$, one has

$$(\hat{V}_{\Lambda_2} - \hat{V}_{\Lambda_1})\varphi = \sqrt{\alpha}\lambda_0 \int_{\mathbb{R}^3} dk \frac{\chi_{\Lambda_2}(k) - \chi_{\Lambda_1}(k)}{|k|} [e^{ik \cdot (-i\nabla_P)} a(k) + e^{-ik \cdot (-i\nabla_P)} a(k)^*] \varphi. \quad (6.15)$$

If $\varphi \geq 0$ w.r.t. \mathfrak{P} , then the right hand side of (6.15) is also positive w.r.t. \mathfrak{P} . Hence one concludes the assertion. \square

Suppose $\Lambda \leq \Lambda'$. For any $\varphi \in \text{dom}(p^2) \cap \text{dom}(N_f) = \text{dom}(\hat{H}_\Lambda) = \text{dom}(\hat{H}_{\Lambda'})$,

$$(\hat{H}_\Lambda - \hat{H}_{\Lambda'})\varphi = (\hat{V}_{\Lambda'} - \hat{V}_\Lambda)\varphi. \quad (6.16)$$

Hence if $\varphi \geq 0$ w.r.t. \mathfrak{P} , the right hand side of (6.16) is positive w.r.t. \mathfrak{P} by Lemma 6.3. Hence one concludes $\hat{H}_\Lambda \supseteq \hat{H}_{\Lambda'}$ w.r.t. \mathfrak{P} . \square

7 Proof of Corollary 5.2

By Theorem 5.1, $\mathcal{F}_x H_\Lambda \mathcal{F}_x^{-1}$ is monotonically decreasing in Λ and uniformly bounded from below. Moreover $e^{-t\mathcal{F}_x H_\Lambda \mathcal{F}_x^{-1}} \supseteq 0$ w.r.t. \mathfrak{P} holds. This means the assumptions **(A. 1)**, **(A. 2)** and (2.3) in Theorem 2.3 are satisfied. Moreover each $\hat{H}_\Lambda(P)$ has the common domain $\mathcal{D} = \text{dom}(p^2) \cap \text{dom}(N_f)$, hence the assumption **(A. 3)** is satisfied as well. Now we can apply Theorem 2.3 and conclude (i) and (ii). \square

8 Proof of Theorem 5.4

8.1 Proof of (i): Uniform lower bound

Choose $0 \leq \Lambda_0 < \infty$ such that $\sqrt{\alpha}\lambda_0 \int dk (1 - \chi_{\Lambda_0}(k))/|k|^4 < 1/8$. Then, by (6.9), for any $\Lambda \geq 0$, one has a uniform lower bound

$$H_\Lambda \geq M_0 \quad (8.1)$$

with $M_0 = -\alpha\lambda_0^2 \int dk \chi_{\Lambda_0}(k)/|k|^2 - 1/2$. On the other hand, by the similar arguments in §5.1, we have

$$U H_\Lambda U^{-1} = \int_{\mathbb{R}^3}^{\oplus} H_\Lambda(P) dP \quad (8.2)$$

with $U = \mathcal{F}_x e^{ix \cdot P_f}$. Hence $\inf_{P \in \mathbb{R}^3} E_\Lambda(P) = E_\Lambda$ which implies $\inf E_\Lambda(P) \geq E_\Lambda$ for almost every P . But since $E_\Lambda(P)$ is continuous in P , this inequality holds true for all $P \in \mathbb{R}^3$. [Proof of the continuity: For all P , $H_\Lambda(P)$ has the common domain $\mathcal{D} = \text{dom}(P_f^2) \cap \text{dom}(N_f)$. Then, for all $\varphi \in \mathcal{D}$, one easily sees $H_\Lambda(P')\varphi \rightarrow H_\Lambda(P)\varphi$ as $P' \rightarrow P$ which implies $H_\Lambda(P')$ converges to $H_\Lambda(P)$ in strong resolvent sense by [37, Theorem VIII. 25].] Combining this with (8.1), we have $H_\Lambda(P) \geq M_0$ for all $P \in \mathbb{R}^3$. \square

8.2 Proof of (ii): Positivity preserving property

We express the Hamiltonian $H_\Lambda(P)$ as

$$H_\Lambda(P) = H_0(P) - W_\Lambda \quad (8.3)$$

with

$$H_0(P) = \frac{1}{2}(P - P_f)^2 + N_f, \quad (8.4)$$

$$W_\Lambda = \sqrt{\alpha}\lambda_0 \int_{|k| \leq \Lambda} dk \frac{1}{|k|} [a(k) + a(k)^*]. \quad (8.5)$$

Lemma 8.1 (Attraction) *For any $\Lambda \geq 0$, $-W_\Lambda$ is attractive in a sense $-W_\Lambda \trianglelefteq 0$ w.r.t. \mathfrak{F}_+ .*

Proof. For any $\Lambda \geq 0$, we have $\sqrt{\alpha}\lambda_0 \int dk \chi_\Lambda(k) a(k)/|k| \trianglelefteq 0$ w.r.t. \mathfrak{F}_+ by Proposition 4.4. Of course its adjoint also preserves the positivity w.r.t. \mathfrak{F}_+ . Thus we conclude the assertion in the lemma. \square

By Proposition 4.3, $e^{-tN_f} \trianglelefteq 0$ w.r.t. \mathfrak{F}_+ . Furthermore $e^{-t(P-P_f)^2} \trianglelefteq 0$ w.r.t. \mathfrak{F} for all P . [Proof: We can write $e^{-t(P-P_f)^2} = e^{-t|P|^2} \oplus \sum_{n \geq 1}^\oplus \exp[-t(P - \sum_{j=1}^n k_j)^2]$. Each n -th component satisfies $\exp[-t(P - \sum_{j=1}^n k_j)^2] \trianglelefteq 0$ w.r.t. $\mathfrak{F}_+^{(n)}$.] This implies $\exp[-tH_0(P)] = \exp[-t\frac{1}{2}(P - P_f)^2] \exp[-tN_f] \trianglelefteq 0$ w.r.t. \mathfrak{F}_+ for all P . Now we can apply Corollary A.2 and conclude $e^{-tH_\Lambda(P)} \trianglelefteq 0$ w.r.t. \mathfrak{F}_+ . \square

8.3 Proof of (iii): Operator monotonicity

First of all, we clarify the monotonicity of the interaction term:

Lemma 8.2 $-W_\Lambda$ is monotonically decreasing in Λ in a sense that

$$\Lambda_1 \leq \Lambda_2 \implies -W_{\Lambda_1} \trianglelefteq -W_{\Lambda_2} \quad \text{w.r.t. } \mathfrak{F}_+. \quad (8.6)$$

Remark 8.3 As before, the attraction becomes stronger, the larger we take the ultra-violet cutoff.

Proof. Suppose that $\Lambda_1 \leq \Lambda_2$. Then, for any $\varphi \in \text{dom}(W_{\Lambda_1}) \cap \text{dom}(W_{\Lambda_2})$, one has

$$(W_{\Lambda_2} - W_{\Lambda_1})\varphi = \sqrt{\alpha}\lambda_0 \int_{\mathbb{R}_k^3} dk \frac{\chi_{\Lambda_2}(k) - \chi_{\Lambda_1}(k)}{|k|} [a(k) + a(k)^*]\varphi. \quad (8.7)$$

Hence if $\varphi \geq 0$ w.r.t. \mathfrak{F}_+ , then the right hand side of (8.7) is positive w.r.t. \mathfrak{F}_+ as well. Hence we obtain the assertion in the lemma. \square

Suppose $\Lambda \leq \Lambda'$. For any $\varphi \in \text{dom}(P_f^2) \cap \text{dom}(N_f) = \text{dom}(H_\Lambda) = \text{dom}(H_{\Lambda'})$, we have

$$(H_\Lambda - H_{\Lambda'})\varphi = (W_{\Lambda'} - W_\Lambda)\varphi. \quad (8.8)$$

Thus if $\varphi \geq 0$ w.r.t. \mathfrak{F}_+ , then the right hand side of (8.8) is also positive w.r.t. \mathfrak{F}_+ by Lemma 8.2. This ompletes the proof of (iii). \square

9 Proof of Corollary 5.5

9.1 Proof of (i) and (ii): Existence of the limit

By Theorem 5.4, $H_\Lambda(P)$ is monotonically decreasing in Λ , uniformly bounded from below, and $e^{-tH_\Lambda(P)} \triangleright 0$ w.r.t. \mathfrak{F}_+ for all $\Lambda \geq 0, P \in \mathbb{R}^3$ and $t \geq 0$. Moreover $H_\Lambda(P)$ has the common domain $\mathcal{D} = \text{dom}(P_f^2) \cap \text{dom}(N_f)$. Thus the assumptions **(A. 1)**, **(A. 2)**, **(A. 3)** and (2.3) are satisfied. Hence we can apply Theorem 2.3 and conclude the existence of the limit $H(P)$ satisfying (i) and (ii). \square

9.2 Proof of (iii): Decomposition of the limit H

For each $\Lambda \geq 0$, one has $UH_\Lambda U^{-1} = \int_{\mathbb{R}^3}^\oplus H_\Lambda(P) dP$. Thus

$$Ue^{-tH_\Lambda}U^{-1} = \int_{\mathbb{R}^3}^\oplus e^{-tH_\Lambda(P)} dP. \quad (9.9)$$

Taking $\Lambda \rightarrow \infty$, one arrives at

$$Ue^{-tH}U^{-1} = \int_{\mathbb{R}^3}^\oplus e^{-tH(P)} dP \quad (9.10)$$

which implies the desired assertion. \square

10 Proof of Theorem 5.6

In the previous work [29], the author proved $e^{-tH(P)} \triangleright 0$ w.r.t. \mathfrak{F}_+ for all $t > 0$ by applying Theorem 2.4. In that proof, a mild ultraviolet cutoff was employed. In this paper, we are treating the sharp cutoff χ_Λ , so that the method in [29] can not be applied directly. In this section, we will provide an alternative proof based on the local ergodicity. This method can cover the case of the sharp ultraviolet cutoff.

To prove Theorem 5.6, we introduce a *local Hamiltonian* $K_\Lambda(P)$ by

$$K_\Lambda(P) = K_0(P) - W_\Lambda \quad (10.1)$$

with

$$K_0(P) = \frac{1}{2}(P - P_{f,\Lambda})^2 + N_{f,\Lambda}, \quad (10.2)$$

$$W_\Lambda = \sqrt{\alpha}\lambda_0 \int_{|k| \leq \Lambda} dk \frac{1}{|k|} [a(k) + a(k)^*]. \quad (10.3)$$

Here

$$P_{f,\Lambda} = \int_{|k| \leq \Lambda} dk k a(k)^* a(k), \quad N_{f,\Lambda} = \int_{|k| \leq \Lambda} dk a(k)^* a(k). \quad (10.4)$$

The following local property is essential for our study.

Lemma 10.1 *Choose $\mu > 0$ such that $H_\Lambda + \mu > 0$ for all $\Lambda \geq 0$. Then, for all $0 \leq \Lambda < \infty$ and $P \in \mathbb{R}^3$, one obtains*

$$Q_\Lambda(H_\Lambda(P) + \mu)^{-1}Q_\Lambda = Q_\Lambda(K_\Lambda(P) + \mu)^{-1}Q_\Lambda. \quad (10.5)$$

Proof. First we remark that, for each $f \in L^2(B_\Lambda)$, $Q_\Lambda a(f) = a(f)Q_\Lambda$ and $Q_\Lambda a(f)^* = a(f)^*Q_\Lambda$ hold. Furthermore since $P_f Q_\Lambda = P_{f,\Lambda} Q_\Lambda = Q_\Lambda P_{f,\Lambda}$ and $N_f Q_\Lambda = N_{f,\Lambda} Q_\Lambda = Q_\Lambda N_{f,\Lambda}$, one sees

$$H_\Lambda(P)Q_\Lambda = Q_\Lambda K_\Lambda(P). \quad (10.6)$$

Thus

$$Q_\Lambda(K_\Lambda(P) + \mu)^{-1} = (H_\Lambda(P) + \mu)^{-1}Q_\Lambda \quad (10.7)$$

holds. This completes the proof. \square

The following proposition was proved in the previous work [29].

Proposition 10.2 (Local ergodicity) *For all $\Lambda > 0, P \in \mathbb{R}^3$ and $t > 0$, one obtains $e^{-tK_\Lambda(P)} \triangleright 0$ w.r.t. $\mathfrak{F}_{\Lambda,+}$.*

Proof of Theorem 5.6

Choose $\varphi, \psi \in \mathfrak{F}_+ \setminus \{0\}$ arbitrarily. Since $\text{s-}\lim_{\Lambda \rightarrow \infty} Q_\Lambda = \mathbb{1}$, there exists a Λ such that $Q_\Lambda \varphi \neq 0$ and $Q_\Lambda \psi \neq 0$. This means $Q_\Lambda \varphi, Q_\Lambda \psi \in \mathfrak{F}_{\Lambda,+} \setminus \{0\}$. Moreover since $Q_\Lambda \geq 0$ and $Q_\Lambda^\perp = \mathbb{1} - Q_\Lambda \geq 0$ w.r.t. \mathfrak{F}_+ by Proposition 4.6, one has $\varphi \geq Q_\Lambda \varphi$ and $\psi \geq Q_\Lambda \psi$ w.r.t. \mathfrak{F}_+ . Note, for μ sufficiently large, $(K_\Lambda(P) + \mu)^{-1} \triangleright 0$ w.r.t. $\mathfrak{F}_{\Lambda,+}$ by Proposition 10.2 and Theorem A.3. Hence one has, by Lemma 10.1,

$$\begin{aligned} \langle \psi, (H_\Lambda(P) + \mu)^{-1} \varphi \rangle &\geq \langle Q_\Lambda \psi, (H_\Lambda(P) + \mu)^{-1} Q_\Lambda \varphi \rangle_{\mathfrak{F}} \\ &= \langle Q_\Lambda \psi, (K_\Lambda(P) + \mu)^{-1} Q_\Lambda \varphi \rangle_{\mathfrak{F}_\Lambda} \quad (\text{By Lemma 10.1}) \\ &> 0. \end{aligned} \quad (10.8)$$

Therefore $\{(H_\Lambda(P) + \mu)^{-1}\}_\Lambda$ is ergodic. Now we can apply Theorem 2.5 and conclude that $e^{-sH(P)} \triangleright 0$ w.r.t. \mathfrak{F}_+ . \square

A Preliminaries

In this section, we will review some preliminary results about the operator inequalities introduced in §2.1. Almost all of results here are taken from the author's previous work [29, 30, 31, 32, 33].

A.1 Operator monotonicity

Proposition A.1 (Monotonicity) *Let A and B be positive self-adjoint operators. We assume the following.*

- (a) $\text{dom}(A) \subseteq \text{dom}(B)$ or $\text{dom}(A) \supseteq \text{dom}(B)$.
- (b) $(A + s)^{-1} \geq 0$ and $(B + s)^{-1} \geq 0$ w.r.t. \mathfrak{p} for all $s > 0$.

Then the following are equivalent to each other.

- (i) $B \geq A$ w.r.t. \mathfrak{p} .

(ii) $(A + s)^{-1} \supseteq (B + s)^{-1}$ w.r.t. \mathfrak{p} for all $s > 0$.

(iii) $e^{-tA} \supseteq e^{-tB}$ w.r.t. \mathfrak{p} for all $t \geq 0$.

Proof. See [29, 30]. \square

Corollary A.2 *Let A be a positive self-adjoint operator and let B be a symmetric operator. Assume the following.*

(i) B is A -bounded with relative bound $a < 1$, i.e., $\text{dom}(A) \subseteq \text{dom}(B)$ and $\|Bx\| \leq a\|Ax\| + b\|x\|$ for all $x \in \text{dom}(A)$.

(ii) $0 \leq e^{-tA}$ w.r.t. \mathfrak{p} for all $t \geq 0$.

(iii) $0 \leq -B$ w.r.t. \mathfrak{p} .

Then $e^{-t(A+B)} \supseteq e^{-tA} \supseteq 0$ w.r.t. \mathfrak{p} for all $t \geq 0$.

Proof. See [29, 30]. \square

A.2 Perron-Frobenius-Faris theorem

Theorem A.3 (Perron-Frobenius-Faris) *Let A be a positive self-adjoint operator on \mathfrak{h} . Suppose that $0 \leq e^{-tA}$ w.r.t. \mathfrak{p} for all $t \geq 0$ and $\inf \text{spec}(A)$ is an eigenvalue. Let P_A be the orthogonal projection onto the closed subspace spanned by eigenvectors associated with $\inf \text{spec}(A)$. Then the following are equivalent.*

(i) $\dim \text{ran} P_A = 1$ and $P_A \triangleright 0$ w.r.t. \mathfrak{p} .

(ii) $0 \triangleleft (A + s)^{-1}$ for some $s > 0$.

(iii) For all $x, y \in \mathfrak{p} \setminus \{0\}$, there exists a $t > 0$ such that $0 < \langle x, e^{-tA}y \rangle$.

(iv) $0 \triangleleft (A + s)^{-1}$ for all $s > 0$.

(v) $0 \triangleleft e^{-tA}$ for all $t > 0$.

Proof. See, e.g., [9, 29, 38]. \square

B A remark on the strongly continuous semigroup

A family of operators $\{T_s \mid 0 \leq s < \infty\}$ on a Hilbert space \mathfrak{h} is called a *one-parameter semigroup* if

(a) $T_0 = \mathbb{1}$,

(b) $T_s T_t = T_{s+t}$ for all $s, t \geq 0$.

In addition if T_s satisfies

(c) for each $x \in \mathfrak{h}$, $s \mapsto T_s x$ is strongly continuous,

then the family is called a *strongly continuous semigroup*.

The following lemma is well-known [37].

Lemma B.1 *Let T_s be a strongly continuous semigroup on a Hilbert space \mathfrak{h} and $Ax = \lim_{s \rightarrow +0} \frac{1}{s}(\mathbb{1} - T_s)x$, where*

$$\text{dom}(A) = \left\{ x \in \mathfrak{h} \mid \lim_{s \rightarrow +0} \frac{1}{s}(\mathbb{1} - T_s)x \text{ exists} \right\}. \quad (\text{B.1})$$

Then A is closed and densely defined.

If we further add conditions on T_s , we can prove the self-adjointness of A as follows.

Proposition B.2 *Let T_s be a strongly continuous semigroup on a Hilbert space \mathfrak{h} . Assume*

(d) *T_s is self-adjoint for all $s \geq 0$.*

(e) *There exists an $M > 0$ such that $\|T_s\| \leq e^{-sM}$ for all $s \geq 0$.*

Then A is self-adjoint, bounded from below by M and $T_s = e^{-sA}$.

Proof. By Lemma B.1 and (d), A is closed and symmetric. We will show $\ker[A^* \pm i] = \{0\}$. (This is equivalent to the self-adjointness of A by the general theorem.) Pick $\eta \in \ker[A^* - i]$. For all $x \in \mathfrak{h}$, one sees

$$\frac{d}{ds} \langle T_s x, \eta \rangle = -\langle A T_s x, \eta \rangle = -\langle T_s x, A^* \eta \rangle = -i \langle T_s x, \eta \rangle. \quad (\text{B.2})$$

Here we used the facts $T_s \text{dom}(A) \subseteq \text{dom}(A)$ and $\frac{d}{ds} T_s x = -A T_s x$. Solving the differential equation, we obtain $\langle T_s x, \eta \rangle = u_0 e^{-is}$. Since

$$|\langle T_s x, \eta \rangle| \leq e^{-sM} \|x\| \|\eta\| \rightarrow 0 \quad (\text{B.3})$$

as $s \rightarrow \infty$, u_0 must be 0. I.e., $\langle x, \eta \rangle = 0$. Since $\text{dom}(A)$ is dense in \mathfrak{h} , this means $\eta = 0$, namely, $\ker[A^* - i] = \{0\}$. Similarly we can show $\ker[A^* + i] = \{0\}$. Thus we have the assertions in the proposition. \square

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